Math 217 Fall 2025
Quiz 8 – Solutions

Dr. Samir Donmazov

1. Complete* the partial sentences below into precise definitions for, or precise mathematical characterizations of, the italicized term:

   (a) Suppose V and W are vector spaces. A *linear transformation* $T : V \to W$ is ...

   > **Solution:** A function $T : V \to W$ such that for all $\mathbf{u}, \mathbf{v} \in V$ and all scalars a, b,
   >
   > $$T(a\mathbf{u} + b\mathbf{v}) = a\,T(\mathbf{u}) + b\,T(\mathbf{v}).$$
   >
   > Equivalently, $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ and $T(a\mathbf{u}) = a\,T(\mathbf{u})$ for all $\mathbf{u}, \mathbf{v} \in V$ and scalars a.

   (b) A *vector* is ...

   > **Solution:** An element of a vector space. More precisely, if V is a vector space over a field $\mathbb{F}$, any $v \in V$ is called a vector (e.g. in $\mathbb{R}^n$, a vector is an ordered n-tuple with the usual addition and scalar multiplication; in $\mathcal{P}_n$, a vector is a polynomial of degree at most n, etc.).

2. Suppose $n, m \in \mathbb{Z}_{>0}$. Recall that $\mathbb{R}^{n\times m}$ denotes the set of $n \times m$ matrices with entries in $\mathbb{R}$. We define $\vec{0} \in \mathbb{R}^{n\times m}$ to be the n by m matrix for which all entries are zero. Show

   VS-4: For all $A \in \mathbb{R}^{n\times m}$ there is a unique element $-A \in \mathbb{R}^{n\times m}$ such that $A + (-A) = \vec{0}$.

> **Solution: Existence.** Define $-A$ entrywise by $(-A)_{ij} = -a_{ij}$. Then $(A + (-A))_{ij} = a_{ij} + (-a_{ij}) = 0$ for all i, j, hence $A + (-A) = \vec{0}$.
>
> **Uniqueness.** If $B \in \mathbb{R}^{n\times m}$ also satisfies $A + B = \vec{0}$, then $B = \vec{0} - A = -A$ (entrywise: $b_{ij} = -a_{ij}$). Therefore the additive inverse is unique.

*For full credit, please write out fully what you mean instead of using shorthand phrases.

3. True or False. If you answer true, then state TRUE. If you answer false, then state FALSE. Justify your answer with either a short proof or an explicit counterexample.

(a) Suppose V is a vector space and $T\colon V \to V$ and $S\colon V \to V$ are linear transformations. The compositions $S \circ T$ and $T \circ S$ are linear maps from V to V *and* $S \circ T = T \circ S$.

Solution: FALSE. While $S \circ T$ and $T \circ S$ are always linear, they need not be equal (linear maps need not commute). For a counterexample, take

$$T(x, y) = (y, 0)^T, \qquad S(x, y)^T = (0, x)^T \quad \text{on } \mathbb{R}^2.$$

Then $(S \circ T)(x, y) = S(y, 0)^T = (0, y)^T$ while $(T \circ S)(x, y)^T = T(0, x)^T = (x, 0)^T$, so $S \circ T \neq T \circ S$.

(b) Suppose $n \in \mathbb{Z}_{>0}$. If $A_1, \ldots, A_k$ are invertible $n \times n$ matrices, then $(A_1 \cdots A_k)^{-1} = A_k^{-1} A_{k-1}^{-1} \cdots A_1^{-1}$.

Solution: TRUE. <u>Base case:</u> For $k = 2$, $(A_1 A_2)^{-1} = A_2^{-1} A_1^{-1}$.

<u>Induction Hypothesis:</u> $(A_1, \ldots, A_{k-1})^{-1} = A_{k-1}^{-1} \ldots A_1^{-1}$. The general statement follows from the base case and induction hypothesis:

$$(A_1 \cdots A_k)^{-1} = \big((A_1 \cdots A_{k-1}) A_k\big)^{-1} = A_k^{-1} (A_1 \cdots A_{k-1})^{-1} = A_k^{-1} \cdots A_1^{-1}.$$